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Quasi-one-dimensional random systems: weak-disorder expansion of Lyapunov exponents

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Abstract. We construct a non-degenerate weak-disorder expansion of Lyapunov exponents of quasi-one-dimensional random systems $M^d \times \infty$, and for both diagonal and off-diagonal disorders. The supersymmetric representation of the Green function enables us to express the coefficients of expansion graphically. For diagonal disorder, an explicit form of expansion up to fourth order in disorder is given. Inside the energy band, there are critical energies for which our expansion does not work. We did not succeed in generalising our expansion for the degenerate case.

1. Introduction

Interest in quasi-one-dimensional random systems has grown rapidly since the work of Pichard and Sarma [1]. They applied the ideas of the finite-size-scaling theory [2] to the problem of Anderson localisation, and found numerically the critical value of disorder for which transitions to the exponentially localised states arise in the band centre of two-dimensional (2D) and three-dimensional (3D) systems. The same method has been used in [3–5].

The analytical treatment of quasi-1D systems has been developed by Pendry [6] who worked out the generalised transfer-matrix method for studying 3D systems. Derrida *et al* [7] derived the weak-disorder expansion (WDE) of Lyapunov exponents (LEs) of quasi-1D system $M \times \infty$. Recently we proposed an alternative method for constructing the WDE of LEs [8], based on the supersymmetric representation of the Green function. The WDE, however, works only for the best LE, i.e., for that which corresponds to the most slowly decreasing term in the Green function. To find all other LE, one has to use another treatment.

The aims of this paper are as follows. (i) To give formulae for calculating all other LEs of the quasi-1D system $M \times \infty$; (ii) to generalise our treatment for non-degenerate systems $M^d \times \infty$, $d = 2, 3, \ldots$; although we hoped [8] that a supersymmetric formalism would enable us to construct also the degenerate WDE for all LEs, we have not succeeded in doing so; (iii) to find the WDE of LEs for systems with both diagonal and off-diagonal disorders; and (iv) to discuss the anomalies in the energy dependency of LE.

In § 2 we derive the Green function for the system $M^d \times \infty$. In § 3 general formulae for the particular sums of LE are presented. The WDE of LE for non-degenerate energies

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is given in § 4. In § 5 we discuss problems that arise in the degenerate cases, and § 6 deals with anomalies that arise in the E-dependency of LEs.

2. Green function

We start with Hamiltonian

$$H = H_0 + H_r \tag{1}$$

where

$$H_0 = \sum_{x,y} |x, y\rangle\langle x+1, y| + |x, y+1\rangle\langle x, y| + t\sum_x |x, M\rangle\langle x, 1| + \text{HC} \tag{2}$$

$$H_r = \lambda \sum_{x,y} e_{xy} |x, y\rangle\langle x, y|. \tag{3}$$

y and $y+1 = 1, 2, \ldots, M$, $x = 1, 2, \ldots$. Here e_{xy} are random independent energies with zero mean value. H_0 corresponds to the motion of an electron in the lattice $M \times \infty$, and t defines the boundary conditions in the y-direction. The energy $\tilde{E}$ of an electron with Hamiltonian H_0 can be separated into two parts

$$\tilde{E} = E_x(n) + E_n \qquad n = 1, 2, \ldots, M. \tag{4}$$

E_n corresponds to the motion in the y-direction and is given as the eigenvalues of the Hamiltonian

$$H_y = \begin{pmatrix} 0 & 1 & & \ldots & & & t \\ 1 & 0 & 1 & \ldots & & & \\ & & & & 1 & 0 & 1 \\ t & & & & & 1 & 0 \end{pmatrix} \tag{5}$$

$$E_n = \begin{cases} 2\cos[\pi n/(M+1)] & t = 0 \\ 2\cos[2\pi n/M] & t = 1. \end{cases} \tag{6}$$

We define the $M \times M$ matrices with elements $Y_{ij}(n)$ as follows:

$$Y_{ij}(n) = \langle i|n\rangle\langle n|j\rangle \tag{7}$$

where $\langle n|i\rangle$ is the eigenvector of H_y corresponding to the eigenenergy E_n. One easily finds the explicit form of $Y_{ij}(n)$:

$$Y_{ij}(n) = \begin{cases} [2/(M+1)]\sin[\pi ni/(M+1)]\sin[\pi nj/(M+1)] & t = 0 \\ (2/M)\sin(2\pi ni/M)\sin(2\pi nj/M) & t = 1. \end{cases}$$

The motion in the x-direction can be characterised by the wavevector q_n which solves the equation

$$2\cos q_n = E_x(n) = \tilde{E} - E_n. \tag{8}$$

In the following we assume $\tilde{E}$ to be complex, possessing the infinitesimally small imaginary part: $\tilde{E} = E + \mathrm{i}0$, and define the sign of the imaginary part of q_n identical to the sign of imaginary part of $\tilde{E}$.

The Green function of an electron with Hamiltonian H_0 reads [10]

$$G^0_{r_0 r}(E + \mathrm{i}0) = \sum_n \tilde{g}_{r_0 r}(n) \tag{9}$$

where $r_0 = (x_0, i)$, $r = (x, j)$, and

$$\tilde{g}_{r_0 r}(n) = Y_{ij}(n) g_{x_0 x}(n) \tag{10}$$

$$g_{x_0 x}(n) = -\mathrm{i} \exp(\mathrm{i} q_n |x - x_0|) Q_n \tag{11}$$

$$Q_n = [4 - (\tilde{E} - E_n)^2]^{-1/2}. \tag{12}$$

Formula (9) can be easily generalised for the system $M^d \times \infty$, $d > 1$. Let us take

$$|n\rangle = |n_1, n_2, \ldots, n_d\rangle \tag{13}$$

$$E_n = E_{n_1} + E_{n_2} + \ldots + E_{n_d} \tag{14}$$

with E_{n_i} given by (6). Let us now renumerate sites $i_1, i_2, \ldots, i_d$ in the cross section perpendicular to the x-direction as

$$i = i_1 + \sum_{\alpha=2}^{d} M^{\alpha-1}(i_\alpha - 1) \tag{15}$$

and define new $M^d \times M^d$ matrices $Y_{ij}(n)$ as follows:

$$Y_{ij}(n) = Y_{i_1 i_2 \ldots i_d, j_1 j_2 \ldots j_d}(n_1, n_2, \ldots, n_d) = \prod_{\alpha=1}^{d} \langle i_\alpha | n_\alpha \rangle \langle n_\alpha | j_\alpha \rangle. \tag{16}$$

The wavevector q_n can again be found from equation (8) with E_n given by (14), and $G^0_{r_0 r}$, $r = (x, i)$ is given in (9) with matrices $\mathbf{Y}$ given by (16). Thus, in what follows all our considerations are valid for any d. Supposing the non-degenerate case, we can order all q such that

$$|\mathrm{Im}\, q_1| < |\mathrm{Im}\, q_2| < \ldots < |\mathrm{Im}\, q_N| \qquad N = M^d. \tag{17}$$

3. Lyapunov exponents

For the one-dimensional case, when the Green function is of the form (11), we can calculate the LE from formulae (9) and (10):

$$\gamma(E + \mathrm{i}0) = \lim_{b-a\to\infty} \frac{1}{b-a} \langle \log G_{ab}(E + \mathrm{i}0) \rangle. \tag{18}$$

For the quasi-1D case, the Green function is an $N \times N$ matrix $\mathbf{G}_{ab}$ and consists of the sum of N exponentially decreasing terms. Thus we are looking for N LEs $\gamma_1, \gamma_2, \ldots, \gamma_N$ with negative real parts. In analogy with (17) we order them as

$$|\mathrm{Re}\, \gamma_1| < |\mathrm{Re}\, \gamma_2| < \ldots < |\mathrm{Re}\, \gamma_N| \tag{17'}$$

(for the zero-disorder case, $\gamma_n(\lambda = 0) = \mathrm{i} q_n$). Then the natural generalisation of (18) is

$$\lim_{b-a\to\infty} \frac{1}{b-a} \langle \log \mathrm{Sp}\, \mathbf{G}_{ab}(E + \mathrm{i}0) \rangle \tag{19}$$

where Sp is the sum of diagonal elements of $\mathbf{G}_{ab}$, and gives only the LE γ_1 with the smallest (in absolute value) real part.

From (9) and (10) one easily finds that

$$\frac{\partial^p \det G^0_{ab}}{\partial g_n^p} = \sum_{\substack{i_1 \ldots i_p \\ j_1 \ldots j_p \\ i_k \neq i_l \\ j_k \neq j_l}} (-1)^{i_1 + \ldots + i_p + j_1 + \ldots j_p} \det_{\substack{i_1 \ldots i_p \\ j_1 \ldots j_p}} Y(n)\, S\det_{\substack{i_1 \ldots i_p \\ j_1 \ldots j_p}} \mathbf{G}^0_{ab}$$

where $\det_{i_1 \ldots i_p, j_1 \ldots j_p} \mathbf{A}$ is determinant of a $p \times p$ matrix consisting of rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_p$ of an $N \times N$ matrix $\mathbf{A}$, and $S\det_{i_1 \ldots i_p, j_1 \ldots j_p} \mathbf{G}^0$ is a determinant of matrix $\mathbf{G}^0$ from which rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_p$ have been omitted.

Owing to (7), for $p > 1$, we have

$$\det_{\substack{i_1 \ldots i_p \\ j_1 \ldots j_p}} \mathbf{Y}(n) \equiv 0 \qquad p > 1 \tag{20}$$

and so

$$\det \mathbf{G}^0_{ab} = g_1 \cdot g_2 \cdot \ldots \cdot g_N.$$

In the same way one can prove that any $\det_{i_1 \ldots i_p, i_1 \ldots i_p} \mathbf{G}^0_{ab}$ is also linear in g_n, and so can be expressed as the sum of the products of p different g_n. Thus, in the limit $|b - a| \to \infty$, in the sum

$$\Sigma^0_p = \sum_{i_1 i_2 \ldots i_p} \det_{\substack{i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p}} \mathbf{G}^0_{ab}$$

only the term proportional to $\exp[\mathrm{i}(q_1 + q_2 + \ldots + q_p)|b - a|]$ survives.

By analogy, if the disorder is taken into account, we obtain

$$\Sigma_p = \sum_{i_1 i_2 \ldots i_p} \det_{\substack{i_1 \ldots i_p \\ i_1 \ldots i_p}} \mathbf{G}_{ab}$$

$$= \sum_{\substack{i_1 \ldots i_p \\ i_k \neq i_l}} \exp[\mathrm{i}(q_{i_1} + \ldots + q_{i_p})|b - a|]\,(\text{power series in } \lambda) \tag{21}$$

and so

$$\Gamma_p = \gamma_1 + \gamma_2 + \ldots + \gamma_p = \lim_{|b-a| \to \infty} \frac{1}{b - a} \langle \log \Sigma_p \rangle. \tag{22}$$

For $p = 1$ one recovers (19).

Let us note that our formalism differs from that commonly used in the literature; the LEs of quasi-1D systems are usually calculated from the product of random matrices. Then there are $2N$ LEs, $\tilde{\gamma}_i = -\tilde{\gamma}_{N+i}$, and, unlike the present formalism, such a method deals with the LEs $\tilde{\gamma}$ with a positive real part. Then the LEs are ordered as

$$\mathrm{Re}\ \tilde{\gamma}_1 > \mathrm{Re}\ \tilde{\gamma}_2 > \ldots > \mathrm{Re}\ \tilde{\gamma}_N$$

and the particular sum

$$\tilde{\Gamma}_p = \tilde{\gamma}_1 + \tilde{\gamma}_2 + \ldots + \tilde{\gamma}_p$$

is calculated [7]. One easily finds a correspondence between γ and $\tilde{\gamma}$ as

$$\gamma_n = -\tilde{\gamma}_{N+1-n}. \tag{23}$$

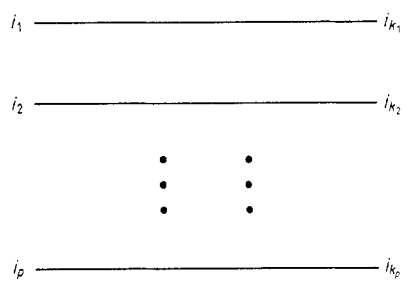

Figure 1. Graphical representation of Σ_p $(k_1, k_2, \ldots, k_p)$ as a permutation ω of $(1, 2, \ldots, p)$. One has to sum over all such permutations with an appropriate factor $P(\omega)$. Owing to equation (28) in the zero-order approximation only a diagonal term with $k_i = i$ survives.

4. Weak-disorder expansion

To construct the WDE of LEs, we first use the replica trick [6, 9, 11] and rewrite (22) in a more suitable form

$$\Gamma_p = \lim_{|b-a|\to\infty} \frac{1}{b-a} \frac{\mathrm{d}}{\mathrm{d}n} \langle \Sigma_p^n \rangle|_{n=0}. \tag{24}$$

Now we express the Green function in the supersymmetric representation [9, 12]

$$G_{\substack{ab \\ ij}}(E + \mathrm{i}0) = -\mathrm{i} \int \mathrm{D}\Phi^* \, \mathrm{D}\Phi \, \xi_{a,i} \xi^*_{b,j} \exp[\mathrm{i}\Phi^*(E + \mathrm{i}0 - H)\Phi] \tag{25}$$

where $\Phi = \binom{\xi}{\rho}$ is supervector consisting of commutative (ξ) and non-commutative vectors (ρ). Inserting (25) into (24), we can average over the disorder, and so obtain the formula

$$\begin{aligned}
\langle \Sigma_p^n \rangle = \sum_{i_1^1 \ldots i_p^1} \cdots \sum_{i_1^n \ldots i_p^n} \sum_{\omega_1 \ldots \omega_n} & P(\omega_1) P(\omega_2) \ldots P(\omega_n) \\
& \times \int \prod_{\alpha=1}^{n} \prod_{\beta=1}^{p} (\mathrm{D}\Phi^{*\alpha\beta} \mathrm{D}\Phi^{\alpha\beta} \xi^{\alpha\beta}_{a i^\alpha_\beta} \xi^{*\alpha\beta}_{b i^\alpha_{k_\beta}}) \\
& \times \exp\left(\sum_{\mu\nu} \mathrm{i}(E + \mathrm{i}0 - H_0)_{\mu\nu} \sum_{\alpha\beta} \Phi^{*\alpha\beta}_\mu \Phi^{\alpha\beta}_\nu \right) \\
& \times \exp\left[-\frac{\lambda^2 \langle e^2 \rangle}{2} \sum_\mu \left(\sum_{\alpha\beta} \Phi^{*\alpha\beta}_\mu \Phi^{\alpha\beta}_\mu \right)^2 \right. \\
& \left. + \frac{\lambda^4}{24} (\langle e^4 \rangle - 3\langle e^2 \rangle^2) \sum_\mu \left(\sum_{\alpha\beta} \Phi^{*\alpha\beta}_\mu \Phi^{\alpha\beta}_\mu \right)^4 - \ldots \right].
\end{aligned} \tag{26}$$

We use the definition of determinant of a $p \times p$ matrix:

$$\det_{\substack{i_1 \ldots i_p \\ i_1 \ldots i_p}} \mathbf{A} = \sum_\omega P(\omega) A_{i_1 i_{k_l}} \ldots A_{i_p i_{k_p}} \tag{27}$$

in which the summation is over all permutations of ω: $(1, 2, \ldots, p) \to (k_1, k_2, \ldots, k_p)$, and $P(\omega)$ defines the corresponding sign: $P(\omega) = \pm 1$.

In (26) the condition $i_k \neq i_l$ for $k \neq l$ is not necessary, since such determinants are evidently zero.

Expanding (26) into power series in λ one obtains the WDE of $\langle \Sigma_p^n \rangle$ which can be easily expressed graphically. To do so, we introduce a 'family'—the graphical representation of Σ_p (figure 1). It contains a sum of products of p Green functions with appropriate

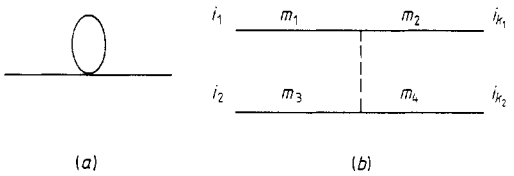

Figure 2. The second-order diagrams of expansion (35). Two-particle diagram (*b*) may correspond to one family if $k_1 = 2$, $k_2 = 1$; then $m_2 = m_3 \neq m_1 = m_4$. For $k_1 = 1$, $k_2 = 2$ lines belong to different families with $m_1 = m_2 = m_3 = m_4$.

indices and signs. Owing to (20) the lines should be represented by the first p terms in sum (9). Since the matrices with elements $Y_{ij}(n)$ are 'orthogonal',

$$\sum_k Y_{ik}(n)Y_{kj}(m) = Y_{ij}(n)\delta_{nm} \tag{28}$$

the zero-order term of each family consists only of the diagonal term with $k_i = i$:

$$\Sigma_p(\lambda = 0) \to g_1 \cdot g_2 \cdot \ldots \cdot g_p \qquad |b - a| \to \infty. \tag{29}$$

The second- and fourth-order terms of the WDE can be expressed by the diagrams presented in figures 2–4. Their contributions can be calculated using the generalised Wick's theorem [9]. Most of the diagrams are known from the WDE of LEs in a 1D system although two differences should be pointed out:

(i) The non-local diagrams in figure 4 depend strongly on the position of the interacting points, and should be calculated separately for $x_1 > x_2, x_1 < x_2$ (i, j arbitrary) and for $x_1 = x_2$ ($i \neq j$).

(ii) While the one-particle diagrams correspond to the first 'diagonal' member in each family, the k-particle diagrams can connect either the lines inside one family or the lines from two or more families.

These features also explain why diagrams 4(*h–j*) survive, and makes the calculation of contributions of diagrams slightly more complicated. In general, one can find the contribution of diagram following the following rules:

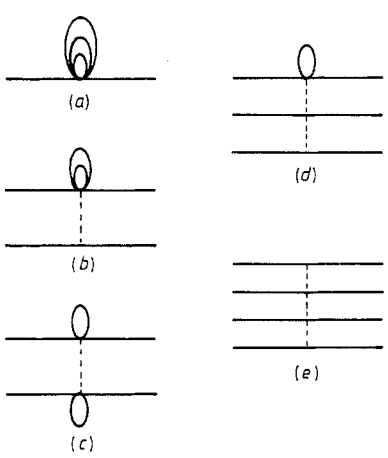

Figure 3. Local (proportional to $\langle e^4 \rangle$) fourth-order diagrams.

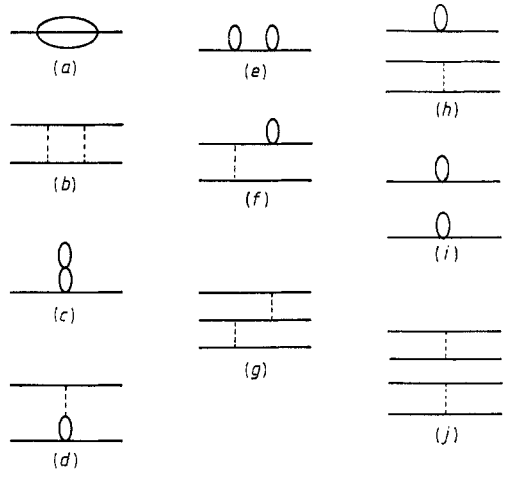

Figure 4. Non-local (proportional to $\langle e^2 \rangle^2$) fourth-order diagrams. Their value strongly depends upon the relations of x-coordinates of interaction points.

(i) the sign of k-particle diagram is $(-1)^{k+1}$.

(ii) The 'inner' lines in diagram (those that do not end in points $x = a$ or $x = b$) may be represented by any factor g_n, $n = 1, \ldots, N$.

(iii) The 'external' lines may be occupied by any g_m, $m = 1, 2, \ldots, p$. Owing to (28), the line starting from i_α and that ending in $i_{k_\beta} = i_\alpha$ are represented by the same g_m.

(iv) Summation over all $i_p = 1, \ldots, M$ and over positions of interacting points should be performed.

As an example, we derive the second-order term. Figure 2(a) has to be diagonal and gives

$$n(b-a)\left(\frac{-\lambda^2\langle e^2\rangle}{2}\right) 2 \sum_{m=1}^{p} \sum_{k=1}^{N} S_{mk}\left[\prod_{\alpha=1}^{p} g_\alpha\right]^n \tag{30}$$

where

$$S_{mk} = \sum_i Y_{ii}(m) Y_{ii}(k) Q_m Q_k. \tag{31}$$

In figure 2(b) we can choose $k_1 = 1$, $k_2 = 2$; then $m_1 = m_2 = m = m_3 = m_4$. Thus the lines belong to different families, and one obtains

$$n(n-1)(b-a)\left(\frac{-\lambda^2\langle e^2\rangle}{2}\right) \sum_{m=1}^{p} S_{mm}\left(\prod_{\alpha_1}^{p} g_\alpha\right)^n. \tag{32}$$

Another contribution is given by $k_1 = 2, k_2 = 1$. Then $m_1 = m_4 = k, m_2 = m_3 = m$. This diagram belongs to one family; owing to (27) it receives the sign (-1) and gives

$$(-1)n(b-a)\left(\frac{-\lambda^2\langle e^2\rangle}{2} \sum_{\substack{m=1 \\ m\neq k}}^{p} \sum_{k=1}^{p} S_{mk}\left(\prod_{\alpha=1}^{p} g_\alpha\right)^n. \right. \tag{33}$$

Expressions (32) and (33) together give the contribution described by rule (iii). From (29)–(33) one finds the second-order approximation of γ_p:

$$\gamma_p = \mathrm{i}q_p - \frac{\lambda^2\langle e^2\rangle}{2}\left(2 \sum_{a=p+1}^{N} S_{pa} + S_{pp}\right). \tag{34}$$

We do not present here the derivation of the fourth-order term, but only give the final formula for Γ_p:

$$\begin{aligned}
\Gamma_p = {} & \mathrm{i}(q_1 + \ldots + q_p) - (\lambda^2\langle e^2\rangle/2)(2S_{ma} + S_{mm}) + (\lambda^4/4)(\langle e^4\rangle - 3\langle e^2\rangle^2) \\
& \times (4T_{m\alpha\beta\gamma} - 6T_{mn\alpha\beta} + 4T_{mnk\alpha} - T_{mnlk}) + (\lambda\langle e^2\rangle^2/4) \\
& \times [(4Z_{m\alpha\beta\gamma} - 6Z_{mn\alpha\beta} + 4Z_{mnk\alpha} - Z_{mnlk}) \\
& + (4X_{m\alpha\beta\gamma} + 4X_{\alpha\beta\gamma m} - 2X_{mn\alpha\beta} - 2X_{\alpha\beta mn} - 8X_{m\alpha n\beta} \\
& + 4X_{mnk\alpha} + 4X_{m\alpha nk} - 2X_{mnlk}) \\
& + (4Z_{m\alpha\beta\gamma}E_{m\alpha\beta\gamma} - 2Z_{mn\alpha\beta}E_{mn\alpha\beta} + 4Z_{\alpha abm} \\
& \times F_{\alpha ab;m}) - 2Z_{abmn}F_{ab;mn} + (4X_{\alpha\beta mn} \cdot E_{\alpha\beta} \\
& + 8X_{\alpha\beta ma} \cdot E_{\alpha\beta} + 4X_{\alpha mab}E_{\alpha m}) + 4X_{mabc}F_{a;m}]
\end{aligned} \tag{35}$$

where

$$T_{abcd} = \sum_i Y_{ii}(a)Y_{ii}(b)Y_{ii}(c)Y_{ii}(d)Q_aQ_bQ_cQ_d \tag{36}$$

$$Z_{abcd} = \sum_{i\neq j} Y_{ij}(a)Y_{ij}(b)Y_{ij}(c)Y_{ij}(d)Q_aQ_bQ_cQ_d \tag{37}$$

$$X_{abcd} = \sum_{ij} Y_{ij}(a)Y_{ij}(b)Y_{ii}(c)Y_{jj}(d)Q_aQ_bQ_cQ_d \tag{38}$$

$$E_{ab\ldots} = \{\exp[-\mathrm{i}(q_a + q_b + \ldots)] - 1\}^{-1} \tag{39}$$

$$F_{ab\ldots;cd\ldots} = \{\exp[-\mathrm{i}(q_a + q_b + \ldots - q_c - q_d - \ldots)] - 1\}^{-1}. \tag{40}$$

In (35), the summation over indices should be performed: $1 \leqslant \alpha, \beta, \gamma \ldots \leqslant N$, $1 \leqslant m, n, l, k \leqslant p$, $p < a, b, c \ldots \leqslant N$. One can easily check that (35) is equivalent to the WDE of [7]. Indeed, the difference

$$\Gamma_{N-p} - \Gamma_N$$

gives, owing to (23), formula (40) of [7].

The graphical expansion can be used also for direct calculation of a given γ_p. To do so, one has to complete contributions of all diagrams in which at least one external line is represented by g_p and all other external lines by g_m, $m \leqslant p$.

4.1. The off-diagonal disorder

As shown in [9], the supersymmetric formalism unifies the problem with diagonal and with off-diagonal disorder. Thus, using techniques developed in § 2–4, one can construct the non-degenerate WDE of all LEs for any system $M^d \times \infty$ with the off-diagonal disorder

$$H_r = \Lambda \sum_{x,y} (v_{xy}|x, y\rangle\langle x + 1, y| + v_{yx}|x, y\rangle\langle x, y + 1|) + \mathrm{HC}.$$

We present here only the expression for γ_p up to second order in the interaction:

$$\gamma_{p,\mathrm{ODD}}(E + \mathrm{i}0) = \mathrm{i}q_p - 2\Lambda^2\langle v^2\rangle\Bigg[\sum_{\alpha=1}^{N}[1 + \exp(\mathrm{i}q_\alpha)\cos q_p]S_{p\alpha}$$

$$- 2\sum_{m=1}^{p}\cos^2\left(\frac{q_p + q_m}{2}\right)S_{pm} + \cos^2 q_p S_{pp}\Bigg]$$

$$- \Lambda^2\langle v^2\rangle\left(\sum_{a=p+1}^{N}\varphi_{pa} + \varphi_{pp}/2\right) \tag{41}$$

where

$$\varphi_{mm} = \sum_{ij}[Y_{ii}(n)Y_{jj}(m) + Y_{ij}(n)Y_{ji}(m)] \tag{42}$$

and the summation in (42) goes over all pairs of the first-neighbour sites i, j in the plane perpendicular to the x-direction.

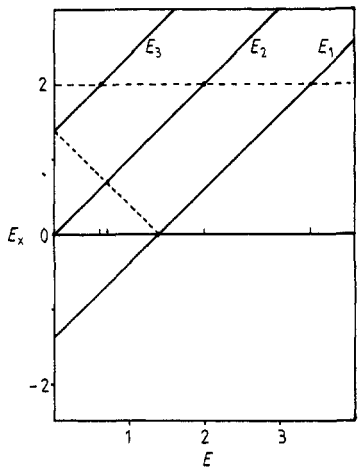

Figure 5. The E-dependence of energies $E_x(n)$ for $M = 3, d = 1$. The critical energies, for which the WDE (35) diverges, are depicted.

5. Degeneracy

Although we have expressed the hope that supersymmetric formalism would enable us to construct the degenerate perturbation theory [8], we have not succeeded in doing so. Degeneracy may arise from the configuration of the system: for $d = 1, t = 1$, or for any $d > 1$ if the boundary conditions in two or more directions are the same. If only $\operatorname{Im} q_r = \operatorname{Im} q_{r+1}$, then equation (35) gives the WDE of Γ_p for $p \leqslant r - 1, p \geqslant r + 1$, but it fails for Γ_r. The reason lies in the fact that non-local diagrams in figure 4 give, in addition to the terms $\sim b - a$, contributions proportional to $(b - a)^2$. For the non-degenerate case, these contributions cancel each other due to the properties of matrices **Y**. For degeneracy $D > 1$, however, it is not true, and so the fourth- and similarly the higher-order terms diverge in the same way as in the WDE of [7]. Thus, we can obtain the WDE of all LEs only for systems with $d = 1, t \neq 1$, or for systems with $d > 1$, if the boundary conditions in directions perpendicular to x differ. If one can remove the degeneracy (e.g., by a small change in boundary conditions), then for $|\operatorname{Im} q_r| - |\operatorname{Im} q_{r+1}| = \Delta = \lambda^2 x$ one finds that the expansion coefficients of γ_r, γ_{r+1} have the form

$$c(x) = c_0 + c_1/x + c_2/x^2 + \dots \tag{43}$$

in agreement with [13].

Another degeneracy—an 'accidental' one—arises in any quasi-1D system in the energy dependence of the LEs. As can be seen from figure 5, there are energies E' in the energy band such that

$$E_x(r) = -E_x(r + 1) \tag{44}$$

and so

$$\operatorname{Im} q_r = \operatorname{Im} q_{r+1} \tag{45}$$

but

$$\operatorname{Re} q_r = \pi - \operatorname{Re} q_{r+1}. \tag{46}$$

In the neighbourhood of E', $E = E' + \lambda^2 x$, one finds expansion coefficients of γ_r, γ_{r+1} of the form (43). Moreover, relation (46) causes the divergencies of all Γ_p.

6. Anomalies

Because all quasi-1D systems are effectively one-dimensional, the WDE suffers from singularities of the same origin as that for the 1D system [9]: there are critical energies inside the energy band for which the coefficients of expansion diverge. As an example, in figure 5 we present the E-dependence of energies $E_x(n) = E - E_n$ for $M = 3$, $d = 1$ (now $E_{1,3} = \pm\sqrt{2}$, $E_2 = 0$). The critical energies, for which the WDE fails, are as follows:

(i) for energies $2 - \sqrt{2}, 2, 2 + \sqrt{2}$ the corresponding $E_x = 2$ and the band edge anomaly arises;

(ii) for $E = 0, 2$ E_x is equal to zero and the band centre anomaly arises;

(iii) the divergency of the fourth-order diagrams may be, in general, reached for energy E which gives a set of solutions q of (9) for which the denominator of any function (39), (40) is zero. The special case of such energies are the accidentally degenerate energies;

(iv) in the higher orders, the anomaly arising in the neighbourhood of energies E such that $E_x(n) = E - E_n = 2\cos(\pi r/s)$ causes the divergency of the $2s$th-order term in the WDE.

In [9] we described the possible regularisation of the WDE of LE in a 1D system, based on the Padé approximation of series like (43). For a diagonal disorder, our approximate formulae differ only slightly from the exact results found in [14–17]. Thus we assume that for diagonal disorders we can also construct similar regularisation for the quasi-1D case. However, in addition to the proof of analyticity of LE in the neighbourhood of critical energies, the calculation of higher-order diagrams is also needed.

7. Conclusion

We presented the non-degenerate weak-disorder expansion of all LEs of an electron in a system $M^d \times \infty$. Our formalism, based on the supersymmetric representation of the Green function, generalises the expansion presented in [7] for the case of arbitrary d and for off-diagonal disorders. For diagonal disorders we presented an explicit form of the WDE of the particular sums of LEs up to fourth order in disorder.

Owing to the large number of critical energies in the energy band we are afraid that the WDE of the form (35) provides insufficient information about the energy dependence of LE even for small disorders. There are non-analyticities in the energy dependence of LEs arising from band centre and band edge anomalies. To treat such anomalies seriously, a special method for calculating the corresponding LEs should be constructed; maybe it would be possible to generalise any method developed for the 1D case [14–17] or self-consistent treatment [18].

In addition to these anomalies, for system $M \times \infty$ there is a large number ($\sim M^2$) of accidentally degenerate energies in the energy band. Thus, the problem of generalisation of expansion (35) for the case of degenerate energies is actual even for the simplest quasi-1D systems. We consider the method presented in [13] to be the most promising to solve this problem.

The situation, similar to that presented in this paper, arises also for the generalised one-dimensional Anderson model [9]. Here the Green function is also given as the sum of exponentially decreasing terms, and so more than one LE should be calculated. We note that similar to the quasi-1D case, the method of calculation of LE presented in [9],

works only for the LE γ_1 with the smallest (in absolute values) real part. The calculation of the particular sums Γ_p of LEs in this case is given elsewhere [19].

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